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Extra-dimensional cosmology with domain-wall branes

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Abstract: We show how to define a consistent braneworld cosmology in a model in which the brane is constructed as a field-theoretic domain wall of finite thickness. The Friedmann, Robertson-Walker metric is recovered in the region of the brane, but, remarkably, with scale factor that depends on particle energy and on particle species, constituting a breakdown of the weak equivalence principle on sufficiently small scales. This unusual effect comes from the extended nature of particles confined to a domain-wall brane, and the fact that they feel an "average" of the bulk spacetime. We demonstrate how to recover the standard results of brane cosmology in the infinitely-thin brane limit, and comment on how our results have the potential to place bounds on parameters such as the thickness of domain-wall braneworlds.

Keywords: p-branes, Large Extra Dimensions, Cosmology of Theories beyond the SM.

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Contents

1. Introduction

Extra-dimensional theories of particle physics beyond the standard model have, in recent years, become a standard part of the array of phenomenological models that we hope to test with the imminent operation of the Large Hadron Collider (LHC). Some models are extended versions of the original Kaluza-Klein (KK) construction, in which the known matter fields propagate in a higher dimensional space, and our usual low-energy physics is augmented by the properties and interactions of their KK towers. More dramatically, inspired by the D-branes of string theory, Arkani-Hamed, Dimopoulos and Dvali [1] (see also [2, 3]) and Randall and Sundrum (RS) [4, 5] have suggested an alternative set of extra dimensional constructions, in which a restricted subclass of fields (often just gravity) propagate in the additional directions, while standard model particles remain confined to a $3+1$ submanifold, or *brane* (see also [6, 7]).

In such models, the brane is typically modelled as infinitely thin — a delta-distribution source — and one then constructs a four-dimensional effective field theory of the standard model particles plus those propagating in the bulk, with their associated KK partners. A rich phenomenology results, with details depending on the particular type of extra dimensional construction employed (see [8 − 11] for alternate constructions, and [12, 13] for general overviews.) Predictions include a spectrum of KK particles expected to be produced at colliders and modifications of the Newtonian inverse square law at sufficiently small distances. Details regarding cosmological evolution were worked out in the initial series of papers [14 − 17], where it was shown that there are deviations from the standard cosmology at temperatures above a particular *normalcy temperature*.

Over the decade following these discoveries, an immense body of work has been performed to tease out the particle physics and cosmological signatures of these models. Cosmology in particular has received much attention [18], with a great deal of the work focused on inflation, gravity perturbations and similar cosmological consequences of braneworlds which would be evident today. Attention has also been given to more exotic phenomena, such as colliding branes [19 – 23], which is a unique aspect of brane cosmology that can have drastic effects on the very early universe. Upcoming precision experiments and missions in these fields, along with particle physics results from the LHC, promise increasingly stringent tests in the near future. Such precision allows us to probe ever more fine detail of extra dimensional theories, requiring an examination of the sensitivity of the tests to the specifics of the underlying construction. One such detail is the delta-distribution nature of the brane itself.

It is certainly possible that the 3-brane on which we find ourselves is an actual D-brane (or orientifold plane), and that our universe is embedded in a string-theoretic realisation of, for example, the Randall-Sundrum braneworld. In that case, it would seem sensible to treat the brane as infinitely thin for all purposes of low-energy particle physics and cosmology. If it is necessary to go beyond this approximation and model the effects of a finite-thickness fundamental brane, then one can do so by averaging the relevant cosmological quantities over the extent of the brane, and imposing boundary matching conditions where the thick brane meets the bulk [24 – 27].

However, there is another possibility, that the braneworld model itself is unrelated to string theory, but is instead a field-theoretic construction, in which the brane is a topological defect (a domain wall, for example) in a higher dimensional space [28, 29]. Generally, a scalar field is responsible for generating the topological defect, and the brane and Randall-Sundrum warped metric become smoothed out versions of their counterparts in the fundamental case [30 – 34]. In such scenarios, the extended nature of the brane in the extra dimensions may have important implications for phenomenology and cosmology, both through its complicated relationship to the bulk, and through the field theory mechanisms through which it must confine the standard model particles.

In a series of papers, two of us (DPG and RRV) have examined this question in detail for the case in which the brane is a codimension one soliton (a domain wall) in $4 + 1$ spacetime dimensions, providing a Minkowski background for standard model fields. In such a construction [35] (see also [36] for an extension to a larger grand-unified group), a subtle interplay between the bulk gauge fields (and gauge group), the bulk scalar fields, and the symmetry breaking field that forms the domain wall is necessary for standard model fermions, the Higgs field, and the $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge bosons to confine and appear in the resulting $3 + 1$ dimensional effective theory.

In this paper, our goal is to extend previous work to understand the cosmology of field-theoretic braneworld models, requiring us to abandon the assumption of Minkowski spacetime on the brane, and to develop a framework in which the standard model fields feel a background Friedmann, Robertson-Walker (FRW) metric. Related work on thick fundamental brane cosmology [25 – 27], and the prescription for averaging 5d quantities to produce the corresponding effective 4d quantities, is drastically modified in the case of a domain-wall brane. As we shall see, the KK expansion of 5d fields and the integration over the full extent of the extra dimension leads to some interesting and unexpected features.

The structure of this paper is as follows. In the next section we review the main points of brane cosmology as constructed around a delta-distribution source. In section 3 we then demonstrate how to build the brane from a field-theoretic domain wall, and discuss how to define the metric on such an object. We then show how scalar fields and fermions are confined to such a domain wall, and derive expressions for the cosmological metrics they experience. The thin wall limit is verified in section 4 and we then collect and discuss our main results.

2. Fundamental brane cosmology

In this section we summarise previous results for the cosmological evolution of a fundamental brane with localised sources (see [14, 15, 17] for details, and [37] for an extension to higher codimension.) The idea is to take a 5-dimensional bulk spacetime, include brane, bulk and brane-localised stress-energy sources, and solve the 5-dimensional Einstein equations. The brane is considered to be a fundamental object and is modelled by a delta distribution, with the total action being

$$\mathcal{S} = \int d^5x\sqrt{-g}\; M_5^3(R-2\Lambda) \;+ \int d^5x\sqrt{-g_{\rm brane}}\;\delta(y)\mathcal{L}_{\rm brane}\,, \tag{2.1}$$

where M_5 is the 5-dimensional gravitational mass scale, Λ is the bulk cosmological constant and g and $g_{\rm brane}$ are the determinants of the metric in the bulk and on the brane respectively. The delta-function localises $\mathcal{L}_{\rm brane}$ to the brane, which includes the brane tension and standard model fields.

Since we are interested in cosmological solutions, we consider sources that are homogeneous and isotropic in the three spatial dimensions, and the most general metric consistent with these symmetries

$$ds_5^2 = -n^2(t,y)dt^2 + a^2(t,y)\gamma_{ij}dx^i dx^j + b^2(t,y)dy^2\,, \tag{2.2}$$

where i,j run over the three spatial dimensions, and γ_{ij} is the metric of the three-space, which may be positively curved, flat or negatively curved. The Einstein equation is

$$R_{MN} - \frac{1}{2}g_{MN}R + g_{MN}\Lambda = \frac{1}{2M_5^3}T_{MN}\,, \tag{2.3}$$

where M,N are 5-dimensional indices. For explicit expressions of the components of the Einstein tensor, see appendix A. The brane tension and brane-localised sources are represented jointly by the 4-dimensional density ρ_b and pressure p_b and appear in the stress-energy tensor as

$$T^M{}_N = \delta(by)\,{\rm diag}(-\rho_b, p_b, p_b, p_b, 0)\;. \tag{2.4}$$

For general values of Λ, ρ_b and p_b, Einstein's equations will yield time-dependent solutions, corresponding, for example, to an expanding spacetime on the brane. Before exploring such solutions, we note that it is possible to fine tune the sources to produce a time-independent background. This corresponds exactly to the scenario of Randall and Sundrum [4, 5] who demonstrated that gravity is localised to a fundamental brane tuned in such a way.

The specific choice necessary is that the brane source be pure tension (corresponding to a 4d cosmological constant) $\rho_b = -p_b = \sigma$, tuned against the bulk cosmological constant according to

$$\sigma = \sqrt{-24M_5^6\Lambda} \,. \tag{2.5}$$

Note that this implies that the bulk geometry must be five-dimensional Anti-de Sitter space AdS_5, with $\Lambda < 0$. The corresponding metric solution is then

$$ds_5^2 = e^{-2\mu|y|}(-dt^2 + \gamma_{ij}dx^i dx^j) + dy^2 \,, \tag{2.6}$$

where

$$\mu \equiv \sqrt{\frac{-\Lambda}{6}} = \frac{\sigma}{12M_5^3} \,. \tag{2.7}$$

Moving back to the general time-dependent case, one solves the 5-dimensional Einstein equations by imposing the Israel matching conditions — calculating the discontinuities in the derivatives of the metric components across $y = 0$, and relating these to the delta-distribution sources (see [14]). It turns out that the behaviours of ρ_b, p_b and the metric components evaluated at $y = 0$ are independent of the metric solutions in the bulk, and obey

$$\dot{\rho}_b + 3H_0(\rho_b + p_b) = 0, \tag{2.8}$$

$$H_0^2 = \frac{\rho_b^2}{144M_5^6} + \frac{\Lambda}{6} - \frac{k}{a_0^2} + \frac{\mathcal{C}}{a_0^4} \,, \tag{2.9}$$

where the parameter k takes values $+1, 0, -1$ according to whether the metric γ_{ij} describes positively-curved, flat, or negatively-curved spatial 3-sections. Here a dot corresponds to a time derivative and time has been rescaled such that $n_0(t) \equiv n(t, y = 0) = 1$. The integration constant $\mathcal{C}$ represents an effective radiation term, or so called "dark radiation" (see [17] for bounds on this term from nucleosynthesis) and from now on we will set $\mathcal{C} = 0$.

The effective 4-dimensional scale factor a_0 is the 5-dimensional metric component $a(t, y)$ evaluated on the brane $a_0(t) \equiv a(t, y = 0)$ and H_0 is the corresponding Hubble parameter. Equation (2.8) describes the usual 4-dimensional conservation of energy on the brane (the continuity equation) and eq. (2.9) is the modified Friedmann equation. Due to the proportionality $H_0 \sim \rho_b$ instead of the usual $H_0 \sim \sqrt{\rho_b}$, this Friedmann equation seems at odds with observation. The clue to fixing this problem comes from considering the time-independent Randall-Sundrum solution, where the brane tension contributed an energy that exactly cancelled the bulk cosmological constant. Guided by this, one writes the total brane source ρ_b, p_b as a sum of a background brane tension σ and some other general brane source ρ, p

$$\rho_b = \sigma + \rho \,, \tag{2.10}$$

$$p_b = -\sigma + p \,, \tag{2.11}$$

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where σ is defined as in (2.5). The effective Friedmann equation for a_0 now reads

$$H_0^2 = \frac{\sigma}{72M_5^6}\rho + \frac{1}{144M_5^6}\rho^2 - \frac{k}{a_0^2} \ . \tag{2.12}$$

If we assume ρ is small compared to σ, then the ρ^2 term gives small corrections to the usual behaviour and the evolution of a_0 is driven to first order by ρ, with the proportionality constant playing the role of the effective Planck mass via

$$M_P^2 \equiv \frac{12M_5^6}{\sigma} = \frac{6M_5^3}{\sqrt{-6\Lambda}} \ . \tag{2.13}$$

An important feature of cosmology in codimension-1 braneworlds is that this entire analysis is independent of the behaviour of the metric components in the bulk. Nevertheless, it is possible to find bulk solutions, and since we will make use of them in a later section we provide them here. For $\mathcal{C} = 0$ they read [17, 13]

$$\begin{aligned} n(t,y) &= e^{-\mu|y|} - \tilde{\epsilon}\sinh(\mu|y|) \, , \\ a(t,y) &= a_0(t)[e^{-\mu|y|} - \epsilon\sinh(\mu|y|)] \, , \\ b(t,y) &= 1, \end{aligned} \tag{2.14}$$

with μ defined as in (2.7) and

$$\epsilon \equiv \frac{\rho}{\sigma} \, , \tag{2.15}$$

$$\tilde{\epsilon} \equiv \epsilon + \frac{\dot{\epsilon}}{H_0} \ . \tag{2.16}$$

Note that for $\epsilon = 0$ and $k = 0$ we recover the RS warped-metric solution given by eq. (2.6).

The parameter ϵ measures the energy density in matter and radiation, relative to the tension of the brane. In terms of this parameter, the Friedmann equation (2.12) is

$$H_0^2 = \frac{1}{6M_P^2}\left(1 + \frac{\epsilon}{2}\right)\rho - \frac{k}{a_0^2} \, , \tag{2.17}$$

demonstrating that $\epsilon \ll 1$ is required to recover conventional cosmology. The earliest direct evidence for our standard cosmological evolution is provided by primordial (big bang) nucleosynthesis (BBN), which takes place at temperatures of order an MeV. Therefore, we are safe from cosmological constraints if we choose $\sigma \gg (1\text{MeV})^4$.

Finally, we briefly discuss the extension of these results to fundamental branes with finite thickness [25 − 27]. In these scenarios the brane and brane-localised sources are modelled as stress-energies distributed over the finite thickness of the brane. The effective 4d quantities, such as the scale factor, energy density and pressure, are defined to be the spatial average, over the extent of the brane in the extra dimension, of their corresponding 5d quantities. One then rewrites Einstein's equations in terms of these averaged quantities and identifies corrections to the infinitely-thin brane scenario. This averaging prescription is an important first step in understanding cosmology away from the infinitely-thin brane limit. However, a more complete treatment is essential, since, for example, in the Minkowski domain-wall set-up, one needs to expand the 5d fields in KK modes and integrate over the full extent of the extra dimension. The rest of this paper is devoted to the development of a more complete averaging framework, within which it is possible to analyse the cosmology of domain-wall brane scenarios.

3. The extension to a domain-wall brane

Our main goal in this paper is to extend the analysis of the previous section to the case in which the brane is topological defect — a domain wall generated by a scalar field. The central problem is how to identify the effective 4-dimensional scale factor (the analogue of a_0) and the equations that describe its time evolution. As we shall see, this question turns out to have an interesting and nontrivial resolution, which may have specific implications for the signatures of such field-theoretic braneworlds.

The creation of a domain-wall brane coupled to gravity is quite straightforward; we will follow closely the construction in [38]. Beginning with a scalar field χ and a suitable potential, boundary conditions are chosen so that χ develops a kink-like profile, which can be thought of as a y-dependent vacuum expectation value. As $y \to \pm\infty$, the value of χ approaches vacuum and its energy density rapidly approaches zero. However, due to the topology of the vacuum (in general a discrete symmetry is required), a domain wall forms around $y = 0$. The combination of gradient and vacuum energy in the core of this object plays an analogous role to the brane tension σ in the fundamental case of the previous section. The shape of the distribution of stress-energy due to the y-dependent profile of χ is a smooth version of the fundamental delta-function brane. In the non-cosmological case, i.e. when one seeks the Minkowski metric on the brane, the solution for the metric then yields a correspondingly smooth version of the $e^{-\mu|y|}$ warp factor in (2.6).

Because this domain-wall brane is extended in the extra dimension y, any fields that were previously brane-localised by the delta function are no longer strictly located at $y = 0$. Rather, such fields (typically the standard model fields) must first be written as full 5d fields, which are coupled to χ in such a way as to produce a Kaluza-Klein tower of 4d fields on the domain wall (in the Minkowski-brane example see e.g. [39]). The ground state profile of the 4d tower has a Gaussian like shape which, when squared[1], reduces to a delta-distribution in the limit of an infinitely-thin domain wall.

Here we are interested in the more general cosmological case. Our objective is to understand the effective 4-dimensional metric on such a domain-wall brane and how the localised fields propagate in that spacetime. In the fundamental-brane case, 4d fields are located at exactly $y = 0$ and have no y degrees of freedom. The 4d metric they feel is thus just the 5d metric evaluated at $y = 0$, and for the RS metric (2.6) this slice is just 4d Minkowski spacetime. For the cosmological metric (2.2) the slice at $y = 0$ has the form

$$ds^2 = -n^2(t, y = 0)dt^2 + a^2(t, y = 0)\gamma_{ij}dx^i dx^j. \tag{3.1}$$

[1] The squaring of the extra-dimensional profile comes from the normalisation of the kinetic term, which is quadratic in the field, hence quadratic in the profile.

By scaling t such that $n(t, y = 0) = 1$, it is clear that the effective 4d metric is of the FRW form, with the effective scale factor defined by $a_{\rm eff}(t) = a_0(t) \equiv a(t, y = 0)$. The solutions to the 5d Einstein equations given in the previous section then describe how $a_{\rm eff}$ evolves, and hence describe the spacetime in which the localised fields propagate. In this fundamental-brane scenario, each field has the same time (with the same normalisation) and feels the same scale factor, and so it is sensible to say that the effective 4d metric is unique and defined by (3.1).

For the domain-wall brane scenario things are quite different and, as we demonstrate explicitly below, we are led to abandon the question "what is the effective scale factor on the brane?", and allow that *different fields may propagate in different spacetimes*. The essential reason for this comes from the extended nature of the profiles, as the associated fields are now sensitive to the metric around $y = 0$, not just the slice exactly at $y = 0$. Since the cosmological evolution of the slices in the vicinity of the brane are misaligned (they expand at different rates), there is a kind of "dimensional parallax" effect, whereby different species of particle are subject to a different averaging (they have a different perspective) of the slices.

We note here that for the Minkowski domain-wall brane with a smoothed-out version of the RS metric (2.6), things are much simpler, because each 4d slice is proportional to Minkowski spacetime. Therefore, the Minkowski part essentially factorises out of the averaging integral and each field feels the same spacetime.

The analysis in section 2 determined the effective scale factor a_0 in the case of a general brane-localised source parameterised by ρ and p. For the domain-wall brane scenario, we need to look at the sources from the more fundamental level of classical fields. The general strategy is to identify the kinetic term in the action for the relevant field, integrate out the extra dimension y, and then to match the resulting 4d effective action to the canonical 4d action for such a field in an FRW background.

3.1 A localised scalar field

We begin by considering a scalar field, turning to fermions in the next subsection. We take the 5d metric given by eq. (2.2) and a 5d scalar field $\Phi(t, x^i, y)$ separated, for reasons we shall expand on below, as $\Phi(t, x^i, y) = f(t, y)\phi(t, x^i)$. The objective is to determine the effective 4d spacetime on which the relevant 4d field ϕ propagates.

The action for a 5d scalar $\Phi(t, x^i, y)$ with metric g_{MN} is

$$\mathcal{S}_5 = \int d^4x \int dy\, \sqrt{g} \left[-\frac{1}{2} g^{MN} \partial_M \Phi \partial_N \Phi - U(\Phi) \right] , \tag{3.2}$$

where the potential U may contain couplings of Φ to the domain-wall field (to localise Φ) or couplings to other fields. Using the metric ansatz (2.2) we then obtain

$$\mathcal{S}_5 = \int d^4x \int dy\, n a^3 b \sqrt{\gamma} \left[-\frac{1}{2} \big(-n^{-2} \dot{\Phi}^2 + a^{-2} \gamma^{ij} \partial_i \Phi\, \partial_j \Phi + b^{-2} (\partial_y \Phi)^2 \big) - U(\Phi) \right] . \tag{3.3}$$

By analogy with the flat case, our first instinct might be to separate variables by writing $\Phi(t, x^i, y) = \sum_n f_n(t, y) \pi_n(x^i)$. However, in the case of a time-dependent metric such an expansion makes it difficult to identify a 4d scalar field, since the time dependence of, for example, a 4d plane wave, is consumed by the profile f_n, and π becomes merely a static spatial wave. The next obvious suggestion is to instead write $\Phi(t, x^i, y) = \sum_n f_n(y)\phi_n(t, x^i)$, so that ϕ_n can be identified as a 4d Kaluza-Klein mode with extra dimensional profile f_n. Here however, we encounter a different problem, namely that the time variation of the metric components implies that the extra-dimensional profile will in general change with time.

We overcome these obstacles by noting that there are really two time scales in the problem: the cosmological time scale of the evolution of the background metric, and the time scale associated with particle physics processes. With this in mind, we consider the following separation of variables

$$\Phi(t, x^i, y) = \sum_n f_n(t, y)\phi_n(t, x^i) \ . \tag{3.4}$$

The possible ambiguity in the time dependence (whether it appears in f_n or ϕ_n) is resolved by the requirement that ϕ_n satisfies the 4d Euler-Lagrange equation, which will be specified below. In order for ϕ_n to be identified as a propagating 4d field, it must also carry the majority of the time dependence, hence we impose the condition $\dot{f}_n/f_n \ll \dot{\phi}_n/\phi_n$. These requirements formally identify the class of solutions for f_n that we are allowing.

One should consider this prescription a separation of scales, rather than a strict separation of variables. Quantitatively, $\dot{\phi}_n/\phi_n \sim E$ where E is the energy of the particle, and $\dot{f}_n/f_n \sim H$ where H is the Hubble constant. In natural units we have $H \sim 10^{-32}$eV, which is tiny compared to the typical energy of a particle. In what follows, we therefore neglect all time derivatives of f_n and of the metric components n, a and b, since they are much smaller than the other terms in the action.

From now on we focus on a single mode of the KK tower and drop the subscript n. Then, with the prescription (3.4) and the assumptions regarding small time-derivatives, the action becomes

$$\mathcal{S}_5 = \int d^4x\sqrt{\gamma}\int dy\left[-\frac{1}{2}\left(-\frac{a^3b}{n}f^2\dot{\phi}^2 + nabf^2\gamma^{ij}\partial_i\phi\,\partial_j\phi + \frac{na^3}{b}f'^2\phi^2\right) - na^3bU\right] , \tag{3.5}$$

where a prime denotes a derivative with respect to y. The third term, proportional to ϕ^2, will contribute to the potential U. Integrating over the extra dimension yields the 4d effective action

$$\mathcal{S}_4 = \int d^4x\sqrt{\gamma}\left[-\frac{1}{2}\left(-F(t)\dot{\phi}^2 + G(t)\gamma^{ij}\partial_i\phi\partial_j\phi\right) + \ldots\right] , \tag{3.6}$$

where we have written only the kinetic terms explicitly, and defined

$$F(t) \equiv \int f^2\frac{a^3b}{n}dy, \qquad\qquad G(t) \equiv \int f^2 nab\, dy. \tag{3.7}$$

The action (3.6) is almost what we are looking for, but what remains is to correctly identify the 4d line element describing the spacetime within which ϕ propagates. To do this, we match to the prototype line element

$$ds_4^2 = -T^2(t)dt^2 + X^2(t)\gamma_{ij}dx^idx^j \ , \tag{3.8}$$

and the corresponding prototype action

$$\mathcal{S}_4^{(\rm proto)} = \int d^4x T(t) X^3(t) \sqrt{\gamma} \left[-\frac{1}{2} \left(-T^{-2}(t)\dot{\phi}^2 + X^{-2}(t)\gamma^{ij}\partial_i\phi\partial_j\phi \right) \right] . \qquad (3.9)$$

Matching the effective action (3.6) with the 4d prototype (3.9) we then obtain

$$F(t) = T^{-1}(t) X^3(t) , \qquad G(t) = T(t) X(t) . \qquad (3.10)$$

Solving for $T(t)$ and $X(t)$ gives

$$T(t) = F^{-1/4}(t) G^{3/4}(t) = \left(\int f^2 \frac{a^3 b}{n} \, dy \right)^{-1/4} \left(\int f^2 nab \, dy \right)^{3/4} , \qquad (3.11)$$

$$X(t) = F^{1/4}(t) G^{1/4}(t) = \left(\int f^2 \frac{a^3 b}{n} \, dy \right)^{1/4} \left(\int f^2 nab \, dy \right)^{1/4} . \qquad (3.12)$$

The time-dependent functions $T(t)$ and $X(t)$ define, along with (3.8), the effective 4d line element followed by the field ϕ. As we shall soon demonstrate, we are free to rescale f by an arbitrary (slowly varying) function of time, and we can use this freedom to fix $T = 1$. This corresponds to choosing a canonical time coordinate. The scale factor for ϕ is then precisely

$$a_\phi(t) = X(t) . \qquad (3.13)$$

Notice that the temporal behaviour of $X(t)$ (and hence a_ϕ) is inherited from the time-dependence of the metric components and possibly $f(t, y)$, all of which are taken to be slowly varying.

This result for the effective scale factor immediately raises two important points. The first is that scalar modes with different profile functions will have different definitions of the scale factor a_ϕ. Thus, it is not possible to define a unique scale factor for this 4d effective theory. Instead, each 4d scalar field, whether it arises from a different 5d field, or is merely a different KK mode of the same 5d field, propagates according to a different effective 4d metric.

The second interesting point is that the procedure above will yield a different result for a fermionic field (and also other spin fields) due to the difference arising from the spin connection in the kinetic term. We will follow this point up in the next section where we explicitly perform the relevant calculation for a fermion.

As a consistency check, we consider eqs. (3.11) and (3.12) in the limit of an infinitely-thin domain wall. In such a limit, the square of a typical ground state profile f becomes proportional to a delta-function distribution, $f^2 \to \delta(by)$. This comes from the kinetic term for ϕ, which is quadratic in f, and must be normalised such that, in the thin brane limit, $\int f^2 d(by) = 1$. The integrals for $T(t)$ and $X(t)$ can then be performed analytically yielding $T(t) = n(t, y = 0)$ and $X(t) = a(t, y = 0)$. These coincide with the fundamental brane case, in which the 4d metric is the 5d metric evaluated on the brane.

A further check on our derivation can be made by looking at the separable (but less general) version of the cosmological metric, given by

$$ds_5^2 = c^2(y) \Big[-dt^2 + \hat{a}^2(t)\gamma_{ij} dx^i dx^j \Big] + \hat{b}^2(y) dy^2 . \qquad (3.14)$$

This ansatz allows for AdS_4 and dS_4 brane solutions, as detailed in [30, 40−46]. The effective 4d metric for ϕ then has $T(t) = (\int f^2 c^2 \hat{b}\; dy)^{1/2}$ and $X(t) = \hat{a}(t)(\int f^2 c^2 \hat{b}\; dy)^{1/2}$. Note that T is constant (f will be time-independent; see later) and we can normalise f to make $T = 1$, and then find that $X(t) = \hat{a}(t)$. In this case we again recover the known result, namely that all fields on the brane feel the same metric.

To complete this formal analysis of Φ we determine the differential equation satisfied by the profile function $f(t, y)$. Our definitions above for the separation of scales ensure that ϕ behaves as a 4d scalar field in a spacetime characterised by $T(t)$ and $X(t)$. This means that ϕ will satisfy the Euler-Lagrange equation

$$-\frac{1}{T^2}\ddot{\phi} + \frac{1}{X^2}\gamma^{ij}\left(\partial_i\partial_j\phi - \Gamma^{(3)\,k}_{\;\;ij}\partial_k\phi\right) = m^2\phi\,, \tag{3.15}$$

where $\Gamma^{(3)\,k}_{\;\;ij}$ are the connection coefficients associated with the 3-space metric γ_{ij} and m is the effective 4d mass of ϕ. The parenthesised term on the left hand side is simply the double covariant-derivative of ϕ with respect to γ_{ij}. Note that we are ignoring time derivatives of $T(t)$ and $X(t)$, which are much smaller that the derivatives of ϕ.

Now consider the 5d Euler-Lagrange equation for Φ

$$g^{MN}\left(\partial_M\partial_N\Phi - \Gamma^{(5)\,P}_{\;\;MN}\partial_P\Phi\right) = \frac{\partial U}{\partial\Phi}\,, \tag{3.16}$$

where $\Gamma^{(5)\,P}_{\;\;MN}$ are the 5d connection coefficients. We first separate variables, neglect time derivatives of n, a, b and f, and use eq. (3.15) to eliminate the spatial derivatives of ϕ (thus the 4d mass will appear). We then linearise the equation, yielding

$$\left[f'' + \left(\frac{n'}{n} + \frac{3a'}{a} - \frac{b'}{b}\right)f' + b^2\left(m^2\frac{X^2}{a^2} - U^{(1)}\right)f\right]\phi + \frac{b^2}{a^2}\left(\frac{X^2}{T^2} - \frac{a^2}{n^2}\right)f\ddot{\phi} = 0\,, \tag{3.17}$$

where $\partial U/\partial\Phi = U^{(1)}\Phi + \mathcal{O}(\Phi^2)$. Notice the appearance of the $\ddot{\phi}$ term, which is absent when we specialise to Minkowski spacetime on the brane. There are two reasons for this. First, there is a mismatch between the 5d ratio of the time and 3-space metric factors, and the corresponding effective 4d ratio; $a^2/n^2 \neq X^2/T^2$. For a Minkowski brane these ratios are equal because each 4d slice of the 5d metric is proportional to Minkowski spacetime. Second, we have employed a separation of scales rather than the usual separation of variables (which did not work in this setting). This $\ddot{\phi}$ term then quantifies the *inability of the domain-wall brane to localise proper 4d effective fields*, at least in a cosmological background.

To proceed, we need to eliminate the ϕ and $\ddot{\phi}$ factors so that we have an equation that can, at least in principle, be used to solve for f. To this end we solve the 4d Euler-Lagrange equation (3.15) (in flat space; $k = 0$) and find "plane waves" of the form

$$\phi(t, x^i) \propto \exp(-iT^2Et + iX^2\gamma_{ij}p^ix^j)\,, \tag{3.18}$$

where E is the energy of the wave, p^i is its momentum, and $T^2E^2 = X^2\gamma_{ij}p^ip^j + m^2$. Then $\ddot{\phi} = -E^2T^4\phi$, and equation (3.17) becomes

$$f'' + \left(\frac{n'}{n} + \frac{3a'}{a} - \frac{b'}{b}\right)f' + b^2\left[m^2\frac{X^2}{a^2} - U^{(1)} - \frac{E^2T^4}{a^2}\left(\frac{X^2}{T^2} - \frac{a^2}{n^2}\right)\right]f = 0\,. \tag{3.19}$$

Usually, such an equation depends only on m, implying that although different masses in the KK tower of 4d fields (ϕ_0, ϕ_1, etc.) have different profiles, these profiles are independent of the energy. Here however, the equation also depends on E, so that quanta with the same mass but different *energies* (or momenta) have different profiles. On the surface, eq. (3.19) looks linear and homogeneous in f, but it is in fact a non-linear integro-differential equation, since both $T(t)$ and $X(t)$ are defined in terms of f. Nevertheless, this equation still has the property that f can be rescaled by a y-independent factor, so long as the eigenvalues m and E are also appropriately rescaled to compensate for the change in $T(t)$ and $X(t)$. In fact, since $\dot{f} \ll E$, as discussed previously, we may even take this factor to have a (mild) time-dependence. As we advertised earlier, the rescaling of f can be used to choose a canonical time coordinate, corresponding to fixing $T = 1$, which is achievable precisely because $T(t)$ depends on f.

As a check on our derivation, for the separable cosmological metric (3.14), the factor $X^2/T^2 - a^2/n^2$ vanishes, and eq. (3.19) simplifies to the known result (see [38])

$$f'' + \left(\frac{4c'}{c} - \frac{\hat{b}'}{\hat{b}}\right) f' + \hat{b}^2 \left[m^2 \frac{1}{c^2} - U^{(1)}\right] f = 0 \ . \tag{3.20}$$

Note the lack of time dependence, implying that f is a function of y only. The profile also no longer depends on the energy of the mode, just its mass, as usual.

Let us summarise our results for scalar fields. Given a 5d background metric, described by functions $n(t, y)$, $a(t, y)$, $b(t, y)$, and a generic coupling potential $U(\Phi)$, we may solve eq. (3.19) for $f(t, y)$. The particular solution depends on a mass eigenvalue m and an energy E. We may then use this solution $f(t, y)$ to determine $T(t)$ and $X(t)$ through eqs. (3.11) and (3.12). What results is the 4d spacetime (described by $T(t)$ and $X(t)$) on which a 4d quantum field ϕ with mass m and energy E propagates. We are free to rescale $f(t, y)$ to impose $T = 1$, so that $X(t)$ can then be interpreted as the effective FRW scale factor. The crucial result to note is that *the scale factor depends on the type of field, its coupling potential, and its 4d mass and momentum.*

3.2 Localised fermions

We now turn to fermions, and perform an analogous calculation to determine the effective scale factor describing the 4-dimensional spacetime on which a localised fermion field propagates.

For a 5d fermion $\Psi(t, x^i, y)$, the action is

$$\mathcal{S}_{5,\Psi} = \int d^4x \int dy \ \sqrt{g} \ \left[\overline{\Psi}\Gamma^A E_A{}^M(\partial_M + \omega_M)\Psi - U_\Psi \overline{\Psi}\Psi\right] \ , \tag{3.21}$$

where Γ^A are the 5d flat-space gamma matrices, $E_A{}^M$ are the vielbeins and ω_M is the spin connection[2]. The gamma-matrices obey $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}$, with $\eta^{AB} = \mathrm{diag}(-1, 1, 1, 1, 1)$. The coefficient U_Ψ of the mass term will in general be a function of other fields, to allow, for example, coupling of the fermion to the domain wall.

[2]We are using A, B to denote 5d flat-space indices and M, N to denote 5d curved-space indices.

As for a scalar field, we perform a separation of scales in time and separation of variables in space[3]

$$\Psi(t, x^i, y) = u(t, y)\psi(t, x^i) \tag{3.22}$$

and expand the kinetic terms, ignoring $\dot{u}$. The action becomes

$$\mathcal{S}_{5,\Psi} = \int d^4x \int dy\; na^3 b\sqrt{\gamma}\; \left[u^2\, \overline{\psi}\left(-n^{-1}\gamma^0\dot{\psi} + a^{-1}\gamma^a {e_a}^j \partial_j \psi\right) + \dots\right], \tag{3.23}$$

where γ^0, γ^a are the 4d flat-space gamma matrices with $\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}$ and ${e_a}^j$ are vielbeins for the 3d space, with γ_{ij} the metric[4]. As before, we require that the correct powers of $n(t, y)$, $a(t, y)$ and $b(t, y)$ match with the relevant terms in the prototype 4d fermion action

$$\mathcal{S}^{(\mathrm{proto})}_{4,\psi} = \int d^4x T_\psi(t) X^3_\psi(t)\sqrt{\gamma}\, \overline{\psi}\left(-T^{-1}_\psi(t)\gamma^0\dot{\psi} + X^{-1}_\psi(t)\gamma^a {e_a}^j \partial_j\psi\right), \tag{3.24}$$

where we have used the prototype line element $ds_4^2 = -T^2_\psi(t)dt^2 + X^2_\psi(t)\gamma_{ij}dx^i dx^j$. Matching kinetic coefficients we obtain

$$F_\psi(t) = \int u^2 a^3 b\, dy = X^3_\psi(t), \tag{3.25}$$

$$G_\psi(t) = \int u^2 n a^2 b\, dy = T_\psi(t) X^2_\psi(t)\,, \tag{3.26}$$

which may be inverted to give

$$T_\psi(t) = F_\psi^{-2/3}(t) G_\psi(t) = \left(\int u^2 a^3 b\, dy\right)^{-2/3}\left(\int u^2 n a^2 b\, dy\right), \tag{3.27}$$

$$X_\psi(t) = F_\psi^{1/3}(t) = \left(\int u^2 a^3 b\, dy\right)^{1/3}. \tag{3.28}$$

These results are similar to the scalar case. As there, we can rescale $u(t, y)$ by a slowly varying function of time to enforce $T_\psi = 1$, so that ψ describes a 4d fermion field in a spacetime with effective scale factor $a_\psi(t) = X(t)$. Again, the definition of the effective scale factor depends on the profile of the particular KK mode ψ that one is interested in. As before, in the infinitely-thin domain-wall limit $u^2 \to \delta(by)$ and $T_\psi(t) \to n(t, y = 0)$, $X_\psi(t) \to a(t, y = 0)$, which recovers the known delta-function brane result. For the separable cosmological metric (3.14), T_ψ is a constant and after normalising u such that $T_\psi = 1$ we have $X_\psi(t) = \hat{a}(t)$: the standard result.

To identify the equation satisfied by $u(t, y)$, we impose the requirement that ψ satisfies the 4d Euler-Lagrange equation with mass m_ψ and use this to eliminate the spatial derivatives of ψ from the 5d Euler-Lagrange equation for Ψ. This yields

$$\left[u' + \left(\frac{n'}{2n} + \frac{3a'}{2a}\right)u\right]\gamma^5\psi + b\left(m_\psi\frac{X}{a} - U_\Psi\right)u\psi - \frac{b}{a}\left(\frac{X_\psi}{T_\psi} - \frac{a}{n}\right)u\gamma^0\dot{\psi} = 0\,, \tag{3.29}$$

[3] There is a subtlety here: we are assuming that all four components of the Dirac spinor ψ have the same profile u, which may not be warranted. We expand on this later.

[4] $\alpha, \beta = 0, 1, 2, 3$ are the 4d flat-spacetime indices, $a = 1, 2, 3$ is a 3d flat-space index.

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where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. This has a similar structure to the scalar version (3.17); in particular the $\dot{\psi}$ term quantifies the deviation from ψ being a 4d field in the usual, infinitely-thin brane definition.

The appearance of γ^0 and γ^5 in eq. (3.29) means localised states on the domain wall have an unusual Dirac structure. In the Minkowski-brane case, the $\gamma^0\dot{\phi}$ term is absent and this leads to the localisation of chiral states, which are eigenspinors of γ^5. With the presence of γ^0, one would naïvely seek Dirac states which are eigenspinors of both γ^0 and γ^5, which is impossible! It therefore seems that the time-dependent background metric leads to unconventionally localised spinor states. To understand this problem more deeply, consider seeking solutions to eq. (3.29) when $\psi = \psi_L$ (and $u = u_L$) is left-chiral, i.e. $\gamma^5\psi_L = -\psi_L$ and $m_{\psi_L} = 0$. Using a plane wave solution for ψ_L and expanding its Weyl components in order to evaluate $\gamma^0\dot{\psi}_L$, we obtain two independent equations for u_L

$$u_L' + \left(\frac{n'}{2n} + \frac{3a'}{2a}\right) u_L + bU_\Psi u_L = 0, \tag{3.30}$$

$$\frac{b}{a}\left(\frac{X_{\psi_L}}{T_{\psi_L}} - \frac{a}{n}\right) u_L E_{\psi_L} T_{\psi_L}^2 = 0\,, \tag{3.31}$$

where E_{ψ_L} is the energy of the chiral plane-wave spinor ψ_L.

With a non-trivial background, the only solution to eq. (3.31) is $u_L = 0$. Thus, there are no localised left-chiral spinors. It may be possible to rectify this problem and find localised states which have a certain definite spinor structure by relaxing the separation ansatz (recall that each component of the Dirac spinor was assumed to have the same profile u, which may be an overly strict assumption), but we will not pursue this line of thought further here.

3.3 The effective Newton's constant

We shall now briefly discuss how to determine the effective Planck mass, and hence Newton's constant, describing the strength of gravity on the brane[5]. Usually, one expands the 5d Ricci scalar in the Einstein-Hilbert action in terms of its 4d counterpart, and the numerical pre-factor is identified as the Planck mass. For example, in the RS2 model, one uses the metric

$$ds_5^2 = e^{-2\mu|y|} g^{(4)}_{\mu\nu}(x^\mu) dx^\mu dx^\nu + dy^2\,, \tag{3.32}$$

for which the Einstein-Hilbert action can be written as

$$\mathcal{S}_{\rm EH} = \int d^4x \int dy \sqrt{-g}\; M_5^3 R \tag{3.33}$$

$$\supset \int d^4x \int dy\; e^{-4\mu|y|} \sqrt{-g^{(4)}}\; M_5^3\; e^{2\mu|y|} R^{(4)}\,, \tag{3.34}$$

where $R^{(4)}$ is the 4d Ricci scalar associated with $g^{(4)}_{\mu\nu}$. One then identifies the effective 4d Planck mass as

$$M_P^2 \equiv M_5^3 \int_{-\infty}^{\infty} e^{-2\mu|y|} dy = \frac{M_5^3}{\mu}\,, \tag{3.35}$$

[5] We are concerned here with the definition of the Planck mass for use in cosmological situations, e.g. in eq. (2.17), as opposed to its use in Cavendish-like experiments.

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which agrees with the result obtained from the effective Friedmann equation in the fundamental brane scenario, eq. (2.13).

Following this approach for domain-wall cosmology, we begin by considering how the 4d Ricci scalar is embedded in the 5d one. However, there is a problem with this approach, because the metric factors for t and x^i behave differently at the 5d level, and so the 5d Ricci scalar does not separate into a 4d piece plus other terms. To make progress, one might consider restricting attention to the 3d Ricci scalar (constructed from the 3d spatial metric γ_{ij}), which does separate, and identifying its pre-factor in the Einstein-Hilbert action as the Planck mass. In other words, we consider just the three spatial components of the metric perturbations to determine the gravitational coupling, instead of using the temporal and spatial components together. Explicitly, we first write the 5d metric in the general form

$$ds_5^2 = -n^2(t,y)dt^2 + a^2(t,y)\xi_{ij}(t,x^i)dx^i dx^j + b^2(t,y)dy^2 \; . \tag{3.36}$$

The 5d Einstein-Hilbert action can then be expanded as

$$\begin{aligned}\mathcal{S}_{\rm EH} &= \int d^4x \int dy\sqrt{-g}\; M_5^3 R &(3.37)\\ &\supset \int d^4x \int dy\; na^3 b\sqrt{\xi}\; M_5^3\; a^{-2}R^{(3)} \,, &(3.38)\end{aligned}$$

where $R^{(3)}$ is the Ricci scalar constructed from ξ_{ij}. The goal is to match this to the 4d prototype Einstein-Hilbert action associated with the general prototype metric

$$ds_4^2 = -T_M^2(t)dt^2 + X_M^2(t)\xi_{ij}(t,x^i)dx^i dx^j \,, \tag{3.39}$$

which yields

$$\begin{aligned}\mathcal{S}_{\rm EH}^{(\rm proto)} &= \int d^4x\sqrt{-g^{(4)}}\; M_P^2 R^{(4)} &(3.40)\\ &\supset \int d^4x T_M(t) X_M^3(t)\sqrt{\xi}\; M_P^2\; X_M^{-2}(t) R^{(3)} \; . &(3.41)\end{aligned}$$

By comparing eqs. (3.38) and (3.41), we can infer that the 5d theory produces 3d (three spatial) metric perturbations with effective Planck mass

$$M_P^2 \equiv \frac{M_5^3}{T_M(t)X_M(t)} \int nab\; dy \; . \tag{3.42}$$

This result requires us to specify the 4d spacetime (by specifying $T_M(t)$ and $X_M(t)$) before we can know the Planck mass. As shown in the previous sub-sections, the 4d spacetime is dependent on the particle species, and so we obtain a *species dependent Planck mass.* This may not be so surprising given that each species follows a different line element, but it is also possible that the assumption of matching only the 3d Ricci scalar is unwarranted.

Perhaps a more sophisticated calculation would try to elucidate the effective 4d Einstein equation, or at least the leading order contribution. Ultimately, we would like to identify $1/2M_P^2$ as the constant of proportionality between the dominant (first order) contributions to the 4d Einstein tensor ${}^{(1)}G_{\mu\nu}^{(4)}$ and the stress-energy tensor ${}^{(1)}T_{\mu\nu}^{(4)}$ for a given field in the thin, large-tension brane limit. One possible way to perform this calculation would be to analyse the equations of motion for metric perturbations. In the case of fundamental-brane cosmology, much of the ground-work for such an analysis has been performed; see for example [47 – 49]. For a domain-wall brane, extra complications arise, again due to the averaging of the metric over the extra dimension. Further, it seems that in order to identify the 4d metric perturbations, one is forced to perform a separation of scales, as in the scalar and fermion case. Such a calculation is beyond the scope of this paper, and for our purposes here we adopt eq. (3.42) as an approximate definition of the effective Planck mass.

4. Effective scale factor for a thin domain wall

Having developed the general framework for a domain wall with localised matter fields and the associated four-dimensional metric, we would like to better understand the behaviour of the effective scale factors a_ϕ and a_ψ. These will, of course, depend on the details of the domain wall construction. Furthermore, we need to solve explicitly for the metric components $n(t,y)$, $a(t,y)$ and $b(t,y)$ in the presence of this domain wall. We are unable to find analytic solutions for a coupled domain-wall gravity system, and numerical solutions are beyond the scope of this initial work. To make progress therefore, we will assume that the domain wall is extremely thin and that therefore the solutions for the metric components are well approximated by the set of equations (2.14).

The profiles of the domain-wall fields will play a role in determining the profiles $f(t,y)$ and $u(t,y)$ of the localised scalars and fermions respectively. These localised fields will then contribute to the total stress-energy and this back-reaction will modify the metric components. However, here we shall ignore such back-reaction effects and consider the thin domain wall as a small perturbation to the fundamental-brane scenario presented in section 2. In order for this perturbative approach to work, it is necessary that the brane localised sources be relatively small, meaning that $\epsilon \ll 1$.

To compute a_ϕ within this approximation scheme, we first normalise $f(t,y)$ such that $T = 1$, by defining

$$f(t,y) = \tau(t)\tilde{f}(t,y) \tag{4.1}$$

so that

$$T(t) = \tau(t)\,\tilde{F}^{-1/4}(t)\,\tilde{G}^{3/4}(t)\,, \qquad X(t) = \tau(t)\,\tilde{F}^{1/4}(t)\,\tilde{G}^{1/4}(t)\,, \tag{4.2}$$

where $\tilde{F}(t)$ and $\tilde{G}(t)$ are defined as in eq. (3.7) but with $f(t,y)$ replaced by $\tilde{f}(t,y)$. Enforcing $T = 1$ gives $\tau(t) = \tilde{F}^{1/4}(t)\,\tilde{G}^{-3/4}(t)$ so that $X(t) = \tilde{F}^{1/2}(t)\,\tilde{G}^{-1/2}(t)$. We may then compute $\tilde{F}(t)$ and $\tilde{G}(t)$ by substituting in for the bulk metric solutions (2.14) yielding, for example,

$$\tilde{G}(t) = \int \tilde{f}\, an\; dy = a_0 \int \tilde{f}\left[e^{-2\mu|y|} - (\epsilon+\tilde{\epsilon})e^{-\mu|y|}\sinh(\mu|y|) + \epsilon\,\tilde{\epsilon}\,\sinh^2(\mu|y|)\right] dy\;. \tag{4.3}$$

Requiring that the localisation profile $\tilde{f}(t,y)$ be sharply peaked at the centre of the domain wall ($y = 0$) and fall off rapidly in the bulk translates to $\tilde{f}^2(t,y)\sinh^2(\mu|y|) \to 0$ as $y \to \pm\infty$.

This condition is consistent with the sufficiently-thin domain-wall brane we are dealing with here. Thus, we may ignore the second order term $\mathcal{O}(\epsilon\,\tilde{\epsilon})$ and write

$$\tilde{G}(t) = a_0 \left[I_1(t) - (\epsilon + \tilde{\epsilon}) I_2(t) \right] \,, \tag{4.4}$$

where

$$I_1(t) = \int \tilde{f}^2(t,y) e^{-2\mu|y|} dy \,, \tag{4.5}$$

$$I_2(t) = \int \tilde{f}^2(t,y) e^{-\mu|y|} \sinh(\mu|y|) dy \,. \tag{4.6}$$

These integrals, $I_1(t)$ and $I_2(t)$, are dependent on the exact form of the extra-dimensional profile $\tilde{f}(t,y)$. However, if the profile is sufficiently peaked, as we are assuming, we have $I_2(t) \ll I_1(t)$, because $\sinh(\mu|y|) \sim 0$ close to the centre of the domain wall. Under these assumptions, we may compute

$$\tilde{F}(t) = a_0^3 \left[I_1(t) - (3\epsilon - \tilde{\epsilon}) I_2(t) \right] \,, \tag{4.7}$$

so that the effective scale factor for the scalar field becomes

$$a_\phi(t) = X(t) = a_0(t) \left[1 - (\epsilon - \tilde{\epsilon}) \frac{I_2(t)}{I_1(t)} \right] \tag{4.8}$$

$$= a_0(t) \left[1 + \frac{\dot{\epsilon}}{H_0} \frac{I_2(t)}{I_1(t)} \right] \,. \tag{4.9}$$

This is one of the main results of our paper — an explicit, quantitative computation of the corrections to the effective 4-dimensional scale factor arising from considering a domain-wall brane, rather than a fundamental one. The corrections are proportional to the ratio between the rate of change of energy density on the brane and the brane tension, and inversely proportional to the effective Hubble parameter. The corrections also depend in a non-trivial way on the specific localisation profile of the associated field, so that different fields are corrected differently.

The expression (4.9) satisfies $a_\phi \to a_0$ for the independent limits of a Minkowski brane with no sources ($\epsilon \to 0$), and an infinitely-thin brane ($I_2 \to 0$). For a concrete example of this latter limit, consider the profile[6]

$$\tilde{f}^2(t,y) = \frac{\Gamma(w+\frac{1}{2})}{\sqrt{\pi}\Gamma(w)} \mu [\cosh(\mu y)]^{-2w} \,, \tag{4.10}$$

which obeys $\tilde{f}^2 \to \delta(y)$ as $w \to \infty$ (the thin domain-wall limit) and is a typical example of smooth localisation factors (see, for example, [39]). It is then straightforward to compute

$$\frac{I_2(t)}{I_1(t)} = \frac{2\Gamma(w+\frac{1}{2}) - \sqrt{\pi}\Gamma(w)}{2\sqrt{\pi}\Gamma(w+1) - 4\Gamma(w+\frac{1}{2})} \quad \xrightarrow{w \to \infty} \quad \frac{1}{\sqrt{\pi w}} \,, \tag{4.11}$$

[6]The time dependence of this sample $\tilde{f}$ would arise from the time dependence of the parameter w, corresponding to the brane thickness changing over time.

which vanishes in the infinitely-thin wall limit. A better approximation for this quantity can be found by solving the differential equation (3.19) for $\tilde{f}(t,y)$, using the background metric components $n(t,y)$ and $a(t,y)$.

For a fermion field the result for the effective scale factor is almost identical to the scalar case

$$a_\psi(t) = a_0(t)\left[1 + \frac{\dot{\epsilon}}{H_0}\frac{J_2(t)}{J_1(t)}\right], \tag{4.12}$$

where the relevant integrals are

$$J_1(t) = \int \tilde{u}^2(t,y)e^{-3\mu|y|}dy, \tag{4.13}$$

$$J_2(t) = \int \tilde{u}^2(t,y)e^{-2\mu|y|}\sinh\mu|y|dy\,, \tag{4.14}$$

and $\tilde{u}(t,y)$ is defined in a similar way to $\tilde{f}(t,y)$.

The results from this section, namely eqs. (4.9) and (4.12), are concrete expressions for modifications to cosmology in a domain-wall brane construction, and are the starting point for an analysis of the constraints on such theories from observations. We expect that a species-dependent scale factor should have an impact on a vast array of cosmological observables, including the predictions of BBN, the era of recombination and the spectra of the microwave background and large scale structure. Acceptable cosmological behaviour should imply constraints on the brane tension σ, which appears in ϵ and μ, and the width of the domain wall, which enters implicitly through the localisation profiles $\tilde{f}$ and $\tilde{u}$.

5. Conclusions

A self-contained and self-consistent construction of a five-dimensional theory requires that the four-dimensional brane on which the standard model fields are confined be formed as a domain wall. Such an object has finite thickness, and couples in a variety of ways to the particle physics and gravitational fields of the theory.

Dimensional reduction from five down to four dimensions involves integrating over the extra dimension in the 5-dimensional action. Each term in the resulting 4-dimensional effective action then contains numerical factors arising from these integrals over the extra-dimensional profiles of the fields. These factors must be absorbed into the fields in order to normalise the kinetic terms. In the case of a Minkowski metric on the brane, this absorption can be carried out via a single definition, to obtain an effective field theory applicable to all fields.

In this paper we have considered the problem of cosmology on a 4-dimensional domain-wall brane. In contrast to the procedure of obtaining a Minkowski metric, this is a complicated process and, in particular, during the normalisation of the kinetic terms, one must take into account the fact that *different fields may feel different spacetimes*. Thus, for domain-wall branes it is not sensible to ask the question "what is the effective scale factor on the brane?". Since the brane has non-trivial dependence on the extra-dimensional coordinate y, and since the metric components $n(t,y)$ and $a(t,y)$ are not proportional to

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each other in the bulk, each 4-dimensional slice at constant y corresponds to a different spacetime. The effective 4-dimensional spacetime for a localised field with a smooth profile in y will thus be an average over all the different slices. This produces a kind of "dimensional parallax" effect, since different fields have different averages, and thus a different "perspective" of the cosmological evolution of each slice.

Therefore, rather than seeking the effective scale factor on the brane (which was possible in the fundamental brane case), we must instead ask: "what is the effective 4d spacetime in which a given localised field propagates?". For each low-energy 4-dimensional field (each species and each mode of the KK tower), we may answer this question by determining the effective 4-dimensional line element ds_4^2. If this line element takes the form of an FRW line element, then we may define an effective scale factor for the associated field. This is as close as we are able to come to answering our original question.

In the case of a localised scalar field, the effective scale factor is given in general by eqs. (3.12) and (3.13), and for fermions one obtains (3.28). For the case of a domain-wall brane which is thin enough such that one can approximate the metric components $n(t, y)$, $a(t, y)$ and $b(t, y)$ with the solutions from the fundamental-brane scenario, the effective scale factors for scalars and fermions are (4.9) and (4.12) respectively. All of these results reduce, in the infinitely-thin domain-wall limit, to the results obtained for fundamental brane cosmology, where the effective scale factor is the bulk metric component $a(t, y)$ evaluated at $y = 0$.

Beyond this basic difference between the fundamental-brane and the domain-wall case, there are a number of other interesting consequences of our construction. In the cosmological scenario, the non-trivial averaging of the metric over y means that, just as different modes of a KK tower have different extra-dimensional profiles, it is also true that different *energies* of the same mass have different profiles. Thus, particles with different energies feel a different scale factor! An outstanding problem is that of defining a unique Planck mass (if possible) at the 4d level. We have provided some insight into the solution to this problem, in the form of the initial approximation (3.42), but a full treatment is beyond the scope of this paper.

The novel features we have presented — species-dependent scale factors and non-unique Planck masses — will lead to potentially observable cosmological phenomena. We are currently performing a detailed phenomenological analysis to determine how these effects constrain parameters such as the width and tension of the domain wall, and the extent to which they may allow new approaches to cosmological problems.

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A. Einstein tensor components

In this appendix we give explicit expressions for the non-zero components of the Einstein tensor

$$G_{MN} = R_{MN} - \frac{1}{2} g_{MN} R \,, \tag{A.1}$$

for the case of our metric ansatz, equation (2.2). The indices 0 and 5 correspond to t and y respectively, while both i and j run over the three spatial dimensions.

$$G_{00} = 3\left[\frac{\dot a}{a}\left(\frac{\dot a}{a}+\frac{\dot b}{b}\right) - \frac{n^2}{b^2}\left(\frac{a''}{a}+\frac{a'}{a}\left(\frac{a'}{a}-\frac{b'}{b}\right)\right)\right], \tag{A.2}$$

$$\begin{aligned} G_{ij} &= \gamma_{ij}\frac{a^2}{b^2}\left[\frac{a'}{a}\left(\frac{a'}{a}+2\frac{n'}{n}\right)-\frac{b'}{b}\left(\frac{n'}{n}+2\frac{a'}{a}\right)+2\frac{a''}{a}+\frac{n''}{n}\right] \\ &\quad + \gamma_{ij}\frac{a^2}{n^2}\left[\frac{\dot a}{a}\left(-\frac{\dot a}{a}+2\frac{\dot n}{n}\right)-2\frac{\ddot a}{a}+\frac{\dot b}{b}\left(-2\frac{\dot a}{a}+\frac{\dot n}{n}\right)-\frac{\ddot b}{b}\right], \end{aligned} \tag{A.3}$$

$$G_{05} = 3\left(\frac{n'}{n}\frac{\dot a}{a}+\frac{a'}{a}\frac{\dot b}{b}-\frac{\dot a'}{a}\right), \tag{A.4}$$

$$G_{55} = 3\left[\frac{a'}{a}\left(\frac{a'}{a}+\frac{n'}{n}\right)-\frac{b^2}{n^2}\left(\frac{\dot a}{a}\left(\frac{\dot a}{a}-\frac{\dot n}{n}\right)+\frac{\ddot a}{a}\right)\right]. \tag{A.5}$$

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